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2016

(July)

MATHEMATICS

(Major)

Paper : IX

(Discrete Mathematics and Functional Analysis)

Full Marks : 90

Time : Three hours

The figures in the margin indicate full marks for the questions.

(A : DISCRETE MATHEMATICS)

1. Answer the following questions : 1×5=5

(a) Distinguish between a sentence and a statement.

(b) Define disjunction with an example.

(c) Give an example of a lattice.

Contd.

(d) What do you mean by size of a graph

(e) How many edges that a complete grap[h] with n vertices can have?

2. Answer the following : 2×5=1[0]

(a) Construct the truth table for

$$(P \vee Q) \vee \rceil P$$

(b) Let $*$ and $\oplus$ are two binary operation[s] of meet and join on a lattice $(L, \leq)$. For any $a, b, c \in L$, show that

(i) $a * (a \oplus b) = a$

(ii) $a \oplus (a * b) = a$

(c) Prove that the number of edges in a graph is equal to half of the sum of degree of vertices.

(d) Draw the logic gates for

(i) $(a+b) \cdot c$

(ii) $\overline{(a+b)}$

(e) What do you mean by a Boolean algebra? Give an example.

3. Answer **any two** of the following : 5×2=10

(a) Let $(L, \leq)$ be a lattice in which $*$ and $\oplus$ denote the operations meet and join respectively. For any $a, b, c \in L$, show that—

(i) $a \leq b \Leftrightarrow a * b = a \Leftrightarrow a \oplus b = b$;

(ii) $b \leq c \Rightarrow a * b \leq a * c$. 3+2=5

(b) Show that the operations of meet and join on a lattice are commutative, associative and idempotent. 5

(c) In any Boolean algebra, show that—

(i) $a = b \Leftrightarrow ab' + a'b = 0$;

(ii) $a = 0 \Leftrightarrow ab' + a'b = 0$;

(iii) $(a+b')(b+c')(c+a') = (a'+b)(b'+c)(c'+a)$;

(iv) $(a+b)(a'+c) = ac + a'b = ac + a'b + bc$;

(v) $a \leq b \Rightarrow a + bc = b(a+c)$ 1×5=5

4. Show that the following are equivalence : 3×2=

(i) $A \to (P \vee C) \Leftrightarrow (A \wedge \neg P) \to C$

(ii) $(P \to C) \wedge (Q \to C) \Leftrightarrow (P \vee Q) \to C$

5. Answer *(a)* and **either** *(b)* **or** *(c)* :

(a) Prove that every simple graph with at least two vertices has two vertices of equal degree. Is the conclusion true for loopless graphs ?

(b) Define path in a graph with example. Prove that a simple connected graph having exactly two vertices which are not cut-vertices is a path. 2+6=8

(c) What do you mean by component? Prove that a simple graph with n vertices and k-components can have at most $(n-k)(n-k+1)/2$ edges. 1+7=8

(B : FUNCTIONAL ANALYSIS)

6. Answer the following : 1×5=5

(a) "Every normed linear space is a metric space." Is the converse true ?

(b) Which of the following statements is true ?

(i) Every normed linear space is a Banach space.

(ii) Every Banach space is a normed linear space.

(c) Define norm of a bounded linear transformation.

(d) What is meant by isometric isomorphism of a normed linear space N into a normed linear space N' ?

(e) Give an example of a normed linear space which is not a Banach space.

7. Answer the following : 2×2=4

(a) In any Hilbert space, prove that

$(x, y+z) = (x, y) + (x, z)$

x, y, z are vectors in the space.

(b) Prove that norm is a continuous function.

8. Answer the following : 3×5=15

(a) When a vector x is said to be orthogonal to a vector y in a Hilbert space ? Give the example of a vector which is orthogonal to itself.

(b) State and prove the Pythagorean theorem for orthogonal vectors in a Hilbert space.

(c) Let N and N' be normed linear spaces and let T be a linear transformation of N into N'. Then show that T is continuous either at every point of N or at no point of N.

(d) Prove that every Hilbert space H which is not equal to zero space possesses an orthonormal set.

(e) In a normed linear space N, with $x, y \in N$, prove that

$$\left| \|x\| - \|y\| \right| \leq \|x - y\|$$

9. Answer **any three** of the following : 5×3=15

(a) Prove that a subspace Y of a Banach space X is complete if and only if the set Y is closed in X.

(b) If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$, then the linear subspace $M + N$ is also closed. Prove it.

(c) Show that the closure of a subspace is also a subspace of a normed linear space.

(d) If S is a closed linear subspace of a Hilbert space H, then prove that

$$H = S \oplus S^{\perp}$$

10. Answer **any one** of the following : 6

(a) Show that the set M of all matrices of the form

$$A_{\alpha, \beta} = \begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$$

where α and β are complex numbers is a Banach space with norm defined by

$$\|A_{\alpha,\beta}\| = |\alpha| + |\beta|$$

(b) Let X and Y be normed linear spaces over the same field and T, a linear mapping of X into Y. Prove that $T : X \to Y$ is continuous on X if and only if T is sequentially continuous on X.