Total number of printed pages–24

3 TDC (Special) MTH M 10

2016

(July)

MATHEMATICS

(Major)

Paper : X

Full Marks : 90

Time : Three hours

The figures in the margin indicate full marks for the questions.

Answer from ***any one*** *Group only*

GROUP—A

(NUMBER THEORY AND LINEAR PROGRAMMING)

Write the answers to the ***two Halves*** *in* ***separate books***

FIRST HALF

(Number Theory)

(*Marks* : 45)

Contd.

1. Answer the following questions : $1\times5=5$

(a) Give a representation of prime numbers which represents infinitely many primes.

(b) How many elements contained in complete set of residue mod m?

(c) Define totally multiplicative arithmetic function.

(d) What is the value of $d(n)$ if n is prime?

(e) What do you mean by pure quadratic congruence?

2. Answer the following questions : $2\times5=10$

(a) Prove that $1 \mid a \ \forall a \in Z$.

(b) If $a, b, k \in Z$, then prove that
$(a+kb, b) = (a, b)$

(c) If $a \equiv b \pmod m$; $c \equiv d \pmod m$, then show that
$$ax + cy \equiv bx + dy \pmod m$$

(d) Find $\sigma\left(p^{\alpha}q^{\beta}\right)$, where p, q are prime.

(e) If a is quadratic non-residue (mod p) and b is quadratic residue (mod p), then ab is quadratic non-residue (mod p). Prove it.

3. Answer ***any three*** questions of the following: $3\times3=9$

(a) Show that $10^n + 3\cdot4^{n+2} + 5$ is divisible by 9.

(b) If $a \equiv b \pmod m$; $a \equiv b \pmod n$, $(m, n) = 1$, then $a \equiv b \pmod{mn}$. Prove it.

(c) Solve for $x, y, z \in \mathbb{N}$
$\phi(x-5) + \phi(3y-5) + \phi(5z-18) = 3$
where ϕ is the Euler function.

(d) If a is quadratic residue (mod p) so is also b if $a \equiv b \pmod p$. Prove it.

4. Answer ***any four*** questions of the following: $4\times4=16$

(a) Prove that the necessary and sufficient condition that the equation $ax + by = c$ will have a solution in integer is $(a, b) \mid c$.

(b) Prove that

$$\sum_{d|n}\mu(d)=\begin{cases}1 & \text{if } n=1\\ 0 & \text{if } n>1\end{cases}$$

(c) Determine :

$$\left[-\frac{42}{31}\right]$$

(d) If (a, b, c) is the solution of $x^2+y^2=z^2$ and $(a, b)=d$, then show that

$$(b, c)=(c, a)=d$$

(e) If (x, y, z) is a primitive solution of $x^2+y^2=z^2$, then prove that one of x and y is even and the other is odd.

5. Answer ***any one*** question of the following : 5

(a) If p is a prime, then show that

$$(p-1)!+1 \equiv 0 \pmod{p}$$

(b) Solve :

$$738x+621y=45$$

SECOND HALF

(Linear Programming)

(*Marks* : 45)

6. Answer the following questions : 1×4=4

(a) Define convex set.

(b) Define basic feasible solution of a linear programming problem.

(c) If the primal problem has feasible solution and the dual has no feasible solution, then what is your opinion regarding the nature of the solution of primal problem ?

(d) What do you mean by a balanced transportation problem ?

7. Answer the following questions : 2×3=6

(a) Transform the following linear programming problem accordingly such that the simplex method may be applicable :

Minimize $Z=2x_1+3x_2-4x_3$

subject to

$$4x_1+2x_2-x_3 \le 4$$
$$-3x_1+2x_2+3x_3 \ge 6$$
$$x_1+x_2-3x_3 = 8$$
$$x_1, x_2, x_3 \ge 0$$

(b) You have given an LPP with five constraints and two variables. Which one of the problems, primal or dual, will you select to solve? Justify your answer.

(c) In a simple loop, how many cells are there in a row? If there are k-such rows, then how many cells are there in the table?

8. (a) Solve by graphical method : 4

Minimize $Z = 2x_1 + 3x_2$

subject to

$$2x_1 + 7x_2 \geq 22$$

$$x_1 + x_2 \geq 6$$

$$5x_1 + x_2 \geq 10$$

$$x_1, x_2 \geq 0$$

(b) Write a short note on least cost method 3

9. (a) Find the initial basic feasible solution of the following problem with the help of North-West corner method and Vogel approximation method and compare their corresponding costs : 6

	D_1	D_2	D_3	D_4	Supply $a_i \downarrow$
O_1	19	30	50	10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
Demand $b_j \rightarrow$	5	8	7	14	34

(b) Solve by simplex method : 6

Maximize $Z = x_1 - x_2 + 3x_3$

subject to

$$x_1 + x_2 + x_3 \leq 10$$

$$2x_1 - x_3 \leq 2$$

$$2x_1 - 2x_2 + 3x_3 \leq 0$$

$$x_1, x_2, x_3 \geq 0$$

10. Answer ***any one*** question of the following : 8

(a) Solving by Big M method, prove that the following LPP has no feasible solution :

Maximize $Z = 2x_1 - x_2 + 5x_3$

subject to

$$x_1 + 2x_2 + 2x_3 \leq 2$$

$$\frac{5}{2}x_1 + 3x_2 + 4x_3 = 12$$

$$4x_1 + 3x_2 + 2x_3 \geq 24$$

$$x_1, x_2, x_3 \geq 0$$

(b) Solve by two-phase method :

Minimize $Z = x_1 + x_2$

subject to

$$2x_1 + x_2 \geq 4$$
$$x_1 + 7x_2 \geq 7$$
$$x_1, x_2 \geq 0$$

11. (a) Show that if any variable of the primal problem be unrestricted in sign, the corresponding dual constraint will be strictly an equation. 4

Or

Find the dual of the problem :

Maximize $Z_x = 2x_1 + 3x_2 - 4x_3$

subject to

$$3x_1 + x_2 + x_3 \leq 2$$
$$-4x_1 + 3x_3 \geq 4$$
$$x_1 - 5x_2 + x_3 = 5$$

and $x_1, x_2 \geq 0$ and x_3 is unrestricted in sign.

(b) Prove that in $\mathbb{R}^2$ the set

$$X = \{(x, y) | x + 3y \leq 5\}$$

is a convex set. 4

Or

Prove that the set of all feasible solutions to an LPP $AX = B, X \geq 0$ is a closed convex set.

GROUP—B

(SPACE DYNAMICS AND RELATIVITY)

Write the answers to the ***two Halves*** *in* ***separate books***

FIRST HALF

(Space Dynamics)

(*Marks* : 45)

1. (a) Define a great circle. 1

(b) State four parts formula in a spherical triangle. 1

2. Answer ***any two*** questions of the following : 4×2=8

(a) In a spherical triangle ABC, prove that $\cos a \cos C = \sin b \cos c - \cos b \sin c \cos A$

(b) Show that the sides and angles of a polar triangle are respectively supplements of the angles and sides of primitive triangle.

(c) If AD is the internal bisector of the angle CAB of the spherical triangle ABC, prove that

$$\cot AD = \frac{1}{2}(\cot b + \cot c)\sec\frac{A}{2}$$

3. (a) Define : 2×2=4

(i) Celestial equator

(ii) Principal vertical circle

(b) Discuss the ecliptical coordinate system.

Or

Discuss the rectangular coordinate system.

(c) If celestial longitude λ and latitude β of a star are known, then find the declination δ of the star.

4. (a) What is the longitude of the sun when it is on the first point of Libra ?

(b) Show that the RA α and the declination δ of the sun will always be connected by the equation $\tan\delta = \tan\varepsilon \sin\alpha$.

Or

If z_1 and z_2 are zenith distances of a star on the observer's meridian and prime vertical respectively, prove that

$$\cot\delta = \operatorname{cosec} z_1 \sec z_2 - \cot z_1$$

where δ is the star's declination.

5. (a) State Kepler's third law. 1

(b) Deduce Kepler's equation. 4

Or

Establish the relation

$$\tan\frac{V}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$$

where V is true anomaly, E is the eccentric anomaly of a planet.

(c) State Newton's law of gravitation. 1

(d) State Kepler's first law and illustrate it. 4

Or

Write a short note on 'two-body' problem.

6. (a) What is artificial satellite ? 1

(b) Give the expression for unperturbed Hamiltonian. 2

(c) Write about the general perturbations of artificial earth satellites. 7

Or

Write the inference of atmospheric density from the motion of satellites.

SECOND HALF

(Special Theory of Relativity)

(*Marks* : 45)

7. *(a)* A reference frame fixed in the earth is an inertial frame. (*State True* **or** *False*) 1

(b) Under what condition Lorentz transformations reduce to Galilean transformation? 1

(c) State the postulates of special theory of relativity. 2

(d) Show that $x^2+y^2+z^2-c^2t^2$ is Lorentz invariant. 4

Or

The length of a rocket ship is 100 *metres* on the ground. When it is in flight, its length observed on the ground is 99 *metres*. Calculate its speed.

8. *(a)* Show that Lorentz transformation equations possess the so-called 'group properties'. 5

Or

Write a short note on : ***(any one)***

(i) Time dilation

(ii) Simultaneity

(b) Derive relativistic formulae for composition of velocities. 5

Or

If u and v are two velocities in the same direction and V their resultant velocity given by

$$\tanh^{-1}\frac{V}{c}=\tanh^{-1}\frac{u}{c}+\tanh^{-1}\frac{v}{c}$$

then deduce the law of composition of velocities from this equation.

(c) Why is the velocity of light called fundamental velocity? 2

9. *(a)* Write the non-relativistic form of Newton's second law. 1

(b) Prove that $E^2=p^2c^2+m_0^2c^4$ is true for all particles in all inertial frames. 3

Or

Calculate the velocity at which the mass of a particle becomes 8 times its rest mass.

(c) Prove that

$$m = \frac{m_0}{\left(1 - \frac{u^2}{c^2}\right)^{1/2}}$$

where u is the velocity of the body when its mass is m and m_0 is the mass of the body when it is at rest. 8

10. (a) What is time-like interval ? 1

(b) Derive relativistic equations of motion 4

(c) Derive the transformation formula for density. 5

Or

Derive the transformation formula for momentum and energy.

GROUP—C

(FINANCIAL MATHEMATICS AND MATHEMATICAL STATISTICS)

Write the answers to the ***two Halves*** *in* ***separate books***

FIRST HALF

(Financial Mathematics)

(*Marks* : 45)

1. Suppose that the supply and demand sets, S and D, for a particular market are described as follows :
S consists of the pairs (q, p) such that $2p-3q=12$ and D consists of the pairs (q, p) such that $2p+q=20$.
Determine the supply function $q^S(p)$, the inverse supply function $p^S(q)$, the demand function $q^D(p)$, the inverse demand function $p^D(q)$ and the equilibrium set. 5

Or

Suppose that the supply and demand functions for a good are
$q^S(p) = bp - a, \qquad q^D(p) = c - dp$
where a, b, c, d are positive constants. Show that the equilibrium price is

$$p^* = \frac{c+a}{b+d}$$

If a tax of T per unit is imposed, find the resulting market price p^T and show that

$$p^T \nleq p^* + T$$

Suppose that the supply and demand functions for a commodity are

$q^S(p) = 3p - 21500, \quad q^D(p) = 8500 - p$

Assuming that the suppliers operate according to the Cobweb model, find a recurrence equation for the sequence p_t of prices. Find the explicit solution given that $p_0 = 7499$, describe how does the sequence p_t behave. 5

Or

The supply and demand functions for a good are

$$q^S(p) = 15p - 41, \quad q^D(p) = 40 - 12p$$

Suppose the suppliers operate according to the Cobweb model and that the initial price is 2·5. Write down explicit formulae for p_t and q_t.

3. The supply and demand sets for a good are

$$S = \{(q, p) \mid q = bp - a\}$$

$$D = \{(q, p) \mid q = c - dp\}; \; a, b, c, d > 0$$

Suppose the government wishes to raise as much money as possible by imposing an excise tax on the good. What should be the value of the excise tax? 5

Or

The only firm manufacturing a certain kind of machine tool can produce upto 100 per week. The demand set for these items is

$$D = \{(q, p) \mid q + 5p = 850\}$$

The cost of producing q items per week is $C(q) = 300 - 10q + q^2$. How many items should be produced each week in order to maximise profit?

4. Answer ***any two*** questions of the following: 5×2=10

(a) The Rhoumbus Corporation is a monopoly with cost function

$$C(q) = q + \frac{2}{100}q^2$$

and the upper limit on its production is 200. The demand set for its product is

$$D = \{(q, p) \mid q + 20p = 300\}$$

Find out the inverse demand function, the profit function, the optimal value q_m and the maximum profit.

(b) Calculate the elasticity of demand when the demand function is given by $q^D(p) = 70 - 4p$. For what range of values of p your expression is valid and for which of these values the demand is inelastic?

(c) Suppose that Robert and Co. is an efficient small firm which cannot produce more than 6 units of its product each week. If their cost function is

$$C(q) = 100 + 20q - 6q^2 + q^3$$

determine the fixed cost, the profit function, the start-up point, the break-even point and the supply set.

5. Answer **any two** questions of the following: 5×2=10

(a) Write a brief note on different types of critical points.

(b) Find the critical points of the function

$$U(x, y) = y^3 + 3xy - x^3$$

$$V(x, y) = x^3 - 3xy^2 + y^4$$

and classify them.

(c) A monopoly manufactures two goods X and Y, with demand functions $x = 12 - p^X$ and $y = 18 - p^Y$. The firm's cost function is

$$C(x, y) = x^2 + y^2 + 2xy$$

Find the maximum profit achievable.

6. Answer **any two** questions of the following: 5×2

(a) What do you mean by Leontief matrix? Show that Leontief matrix is invertible.

(b) A factory makes two goods, grommets and widgets. To make Re 1 worth of grommets requires Re 0·2 worth grommets and Re 0·1 worth of widgets and to make Re 1 worth of widgets requires Re 0·05 worth of grommets and Re 0·1 worth of widgets. There is a market demand of Rs 750 worth of grommets and Rs 500 worth of widgets. What should the total production of each be to meet the market demand?

(c) Suppose that an investor invests his money in three different assets and that three possible states can occur. Show that if the returns matrix is

$$R = \begin{pmatrix} 0{\cdot}95 & 0{\cdot}9 & 1{\cdot}0 \\ 1{\cdot}1 & 1{\cdot}1 & 1{\cdot}1 \\ 1{\cdot}2 & 1{\cdot}15 & 1{\cdot}25 \end{pmatrix}$$

then there is no vector of state prices. Show that

$$Y = (1000 \quad -5000 \quad 4000)$$

$$Z = (\ 0 \quad -5000 \quad 5000)$$

are arbitrage portfolios. Which of the two would you choose, given the choice?

SECOND HALF

(Mathematical Statistics)

(*Marks* : 45)

7. *(a)* State the difference between classification and tabulation.

(b) Answer ***either*** *(i)* ***or*** *(ii)* :

(i) State the objectives of tabulation of data.

(ii) State the merit of diagrammatic representation of data.

8. *(a)* Show that the mean of a variable whose given values are all equal must also be the same as their common value.

(b) Answer ***either*** *(i)* ***or*** *(ii)* :

(i) Derive median and mode graphically.

(ii) Explain the uses of arithmetic mean, median and mode.

9. *(a)* Which measure of dispersion is affected by extreme values ?

(b) Answer ***either*** *(i)* ***or*** *(ii)* :

(i) Define standard deviation. Find standard deviation from the following data : 1+3=4

5, 8, 7, 11, 9, 10, 8, 2, 4, 6

(ii) Define coefficient of variation. Find coefficient of variation from the following data : 1+3=4

Class	:	50-100	100-150	150-200	200-250
Frequency	:	5	8	9	12
Class	:	250-300	300-350	350-400	
Frequency	:	18	23	17	

10. *(a)* State True ***or*** False : 1

The correlation coefficient is a number in [–1, 1] and zero when the variables are independent.

(b) Answer ***either*** *(i)* ***or*** *(ii)* : 4

(i) Show that correlation coefficient is independent of change of origin and scale.

(ii) Show that the standard errors of estimate of X and Y are respectively

$$S_X = \sigma_X \left(1 - r^2\right)^{1/2}$$

$$S_Y = \sigma_Y \left(1 - r^2\right)^{1/2}$$

where the symbols have their usual meanings.

11. *(a)* Define time series.

(b) Answer ***either*** *(i)* ***or*** *(ii)* :

(i) Describe moving average metho for measuring trend, together wi its advantages.

(ii) Describe method of least squa for measuring trend, together wi its advantages.

12. *(a)* State True ***or*** False :

The binomial distribution is continuous distribution.

(b) Comment on the validity of the followi statement :

The mean of a binomial distribution 3 and variance is 4.

(c) Answer ***either*** *(i)* ***or*** *(ii)* :

(i) In a Poisson distributio probability corresponding to successes is $\frac{2}{3}$ times t probability corresponding to successes. Find the mean a standard deviation.

(ii) If X is a Poisson variate s.t.

$P(X=2) = 9P(X=4) + 90P(X=6)$

find the mean of X.

(d) Answer ***either*** *(i)* ***or*** *(ii)* :

(i) When is a random variable said to have normal distribution? Show that for a normal distribution

Mean = Median 1+3=4

(ii) If X is a normal variate with mean 30 and standard deviation 5, find the probability that —

(1) $26 \leq X \leq 40$;

(2) $X \geq 45$. 2+2=4

13. *(a)* State what is meant by test of statistical hypothesis. 1

(b) Answer ***either*** *(i)* ***or*** *(ii)* : 4

(i) Describe what do you mean by critical region.

(ii) Explain the following terms in brief:

(1) Null hypothesis

(2) Alternative hypothesis

(3) Simple hypothesis

(4) Composite hypothesis

(c) Answer **either** *(i)* **or** *(ii)* :

(i) Describe the conditions for valid of chi-square test.

(ii) Give a description on chi-squa probability curve with necessa diagram.