**5 SEM TDC MTH M 2**

**2018**

( November )

MATHEMATICS

( Major )

Course : 502

**( Linear Algebra and Number Theory )**

*Full Marks* : 80

*Pass Marks* : 32/24

*Time* : 3 hours

*The figures in the margin indicate full marks for the questions*

GROUP—A

**( Linear Algebra )**

( *Marks* : 40 )

1. *(a)* Choose the correct answer from the brackets to fill in the blank : 1

"A set consisting of a single non-zero vector is ______."

( linearly dependent / linearly independent )

(b) For $x = (x_1, x_2)$ and $y = (y_1, y_2)$ of $\mathbb{R}^2$ and $\alpha \in \mathbb{R}$, let $x + y = (x_1 + y_1, x_2 + y_2)$ and $\alpha x = \alpha (x_1, x_2) = (\alpha x_1, 0)$.

Is $\mathbb{R}^2$ a vector space with respect to the above operations? Justify your answer. 1+1=

(c) Let $V$ be a vector space and $X$ be a non-empty set. Let $W$ be the set of functions $f : X \to V$. On $W$, define addition and scalar multiplication as follows

$$(f + g)(x) = f(x) + g(x),\ f, g \in W,\ x \in X$$

$$(\alpha \cdot f)(x) = \alpha f(x),\ \alpha \in \mathbb{R},\ x \in X$$

Then show that $W$ is a vector space.

(d) Prove that any basis of a finite dimensional vector space is finite.

(e) Let $W$ be the subspace of $\mathbb{R}^4$ generated by the vectors $(1, -2, 5, -3)$, $(2, 3, 1, -4)$ and $(3, 8, -3, -5)$. Find a basis of $W$.

(f) Find for what value of $k$ the vector $u = (1, -2, k)$ in $\mathbb{R}^3$ is a linear combination of the vectors $v = (3, 0, -2)$ and $w = (2, -1, -5)$.





(g) Let $W_1$ and $W_2$ be two subspaces of a finite dimensional vector space, then show that

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$ 4

2. (a) Let $V$ be a finite dimensional vector space, then prove that any two bases of $V$ have the same number of elements. 4

(b) Examine whether the following mappings are linear or not : 2+2=4

(i) $T : \mathbb{R}^3 \to \mathbb{R}$ defined by

$$T(x, y, z) = 2x - 3y + 4z$$

(ii) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by

$$T(x, y, z) = (|x|, 0)$$

(c) Let $S$ and $T$ be two subsets of a vector space $V(F)$, then show that

$$S \subseteq L(T) \Rightarrow L(S) \subseteq L(T)$$ 4

(d) Define line of a vector space. Show that any two distinct points determine a unique line. 1+2=3

(e) Prove that any two $n$-dimensional vector spaces are isomorphic. 5

## GROUP—B

## ( Number Theory )

( *Marks* : 40 )

**3.** For any integers $a$, $b$, $c$, show that $a|b$, $a|c \Rightarrow a|bx+cy \ \forall x, y \in \mathbb{Z}$.

**4.** Answer any *two* from the following : 3×2=6

(a) Let $a$ and $b$ any positive integers, then prove that $(a, b)\cdot[a, b] = ab$.

(b) Using Euclidean algorithm, solve the equation $726x + 275y = 11$.

(c) Prove that there is no natural number in between 0 and 1.

**5.** (a) Show that there are infinitely many primes of the form $4n+3$.

(b) Prove that

$$S_n = \frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}$$

is never an integer.

(c) Let $p$ be a prime and $a$ any integer, then show that either $p|a$ or $(a, p) = 1$.



( Continued



**6.** (a) State Fermat's little theorem. 1

(b) Show that the relation of congruence on integers is an equivalence relation. 3

(c) Show that $10^n + 3\cdot 4^{n+2} + 5$ is divisible by 9. 3

(d) If $p$ is prime, then prove that 4

$$(p-1)! \equiv -1 \pmod p$$

(e) Find all solutions in positive integers of 4

$$5x + 3y = 52$$

*Or*

Using Chinese Remainder theorem, solve

$$x \equiv 5 \pmod{18}$$
$$x \equiv -1 \pmod{24}$$
$$x \equiv 17 \pmod{33}$$ 4

**7.** (a) Prove that 3

$$P(n) = n^{\frac{d(n)}{2}}$$

(b) Prove that 4

$$\sum_{d|n} \phi(d) = n, \quad \forall n \in \mathbb{N}$$

(c) What are the positive integers $x$, $y$ that satisfy the expression $\phi(xy) = \phi(x) + \phi(y)$? 3

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